Computing Generalized Rank Via Zigzag Persistence and its Applications

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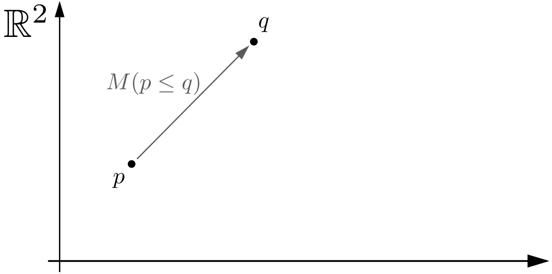
2-parameter persistence and rank invariant



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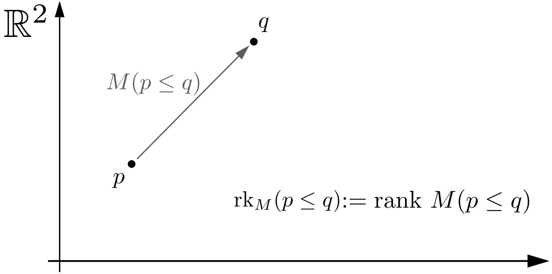

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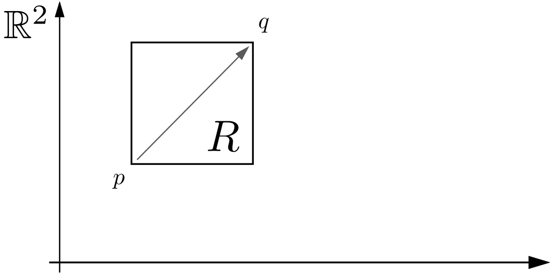

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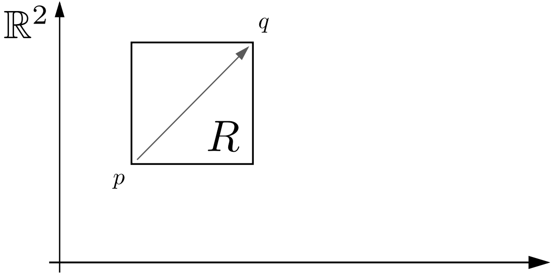

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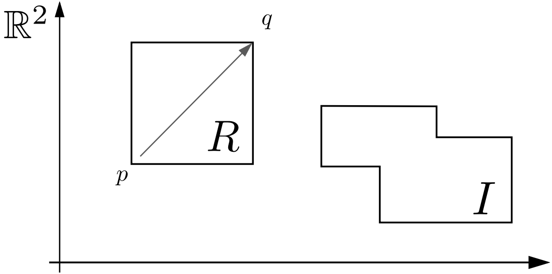

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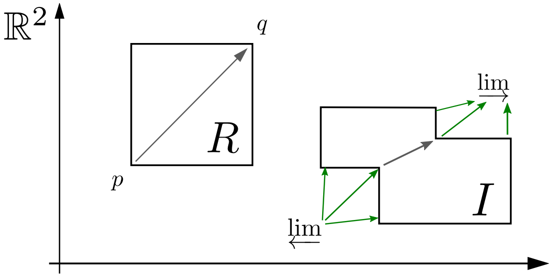

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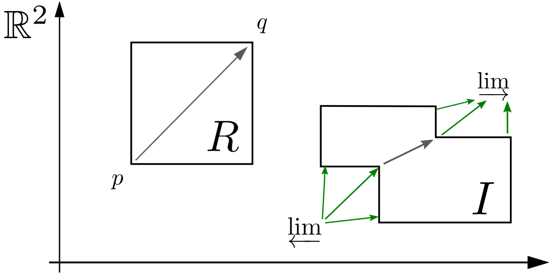

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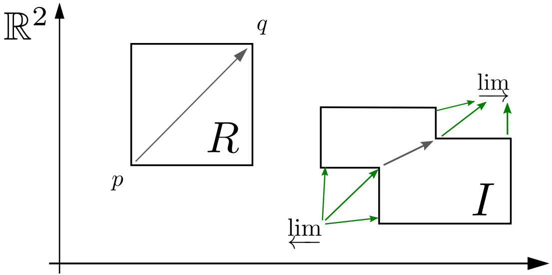

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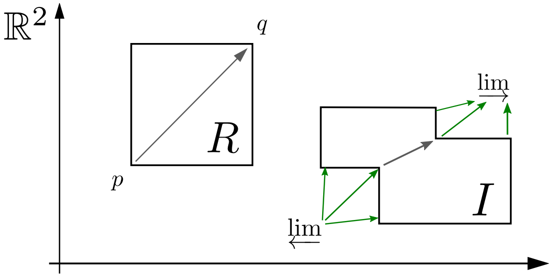

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How to compute the generalized rank invariant?

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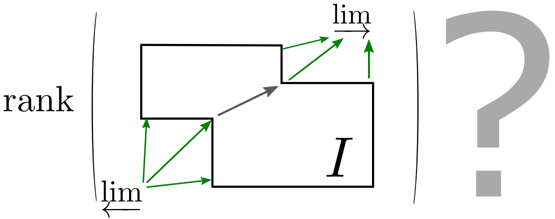
How to compute



Main theorem

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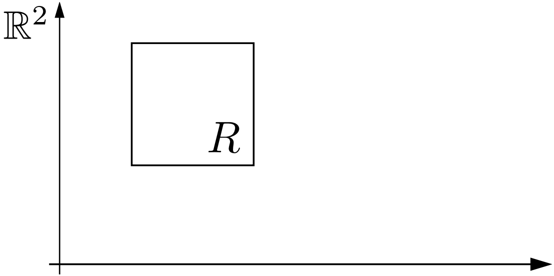

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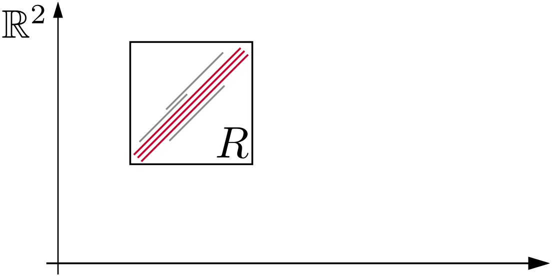

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$$R \mapsto \textbf{rk}_M(R)$$
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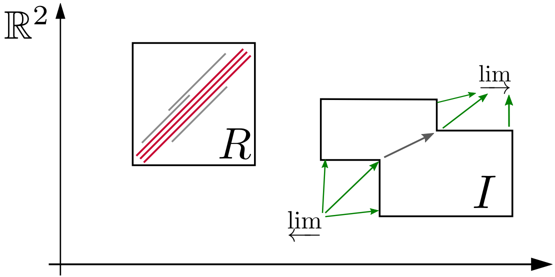

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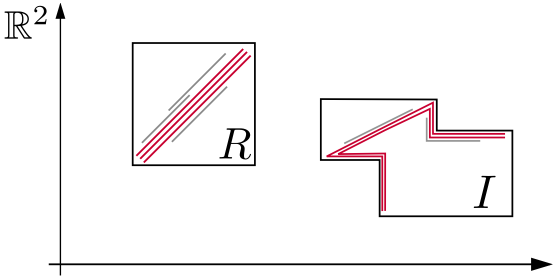

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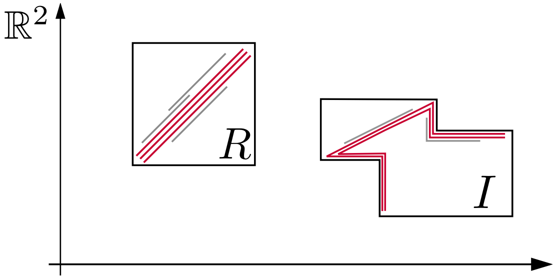

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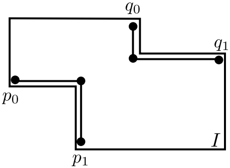

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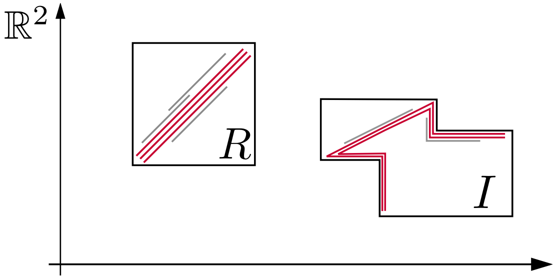

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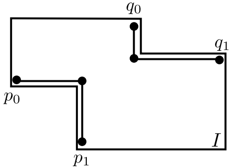

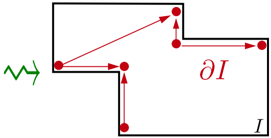

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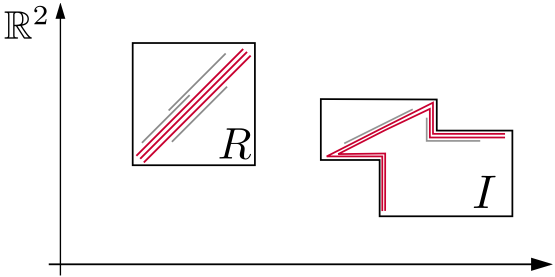

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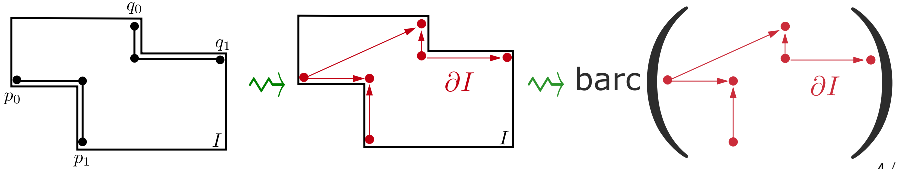

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- an arrow $\leftarrow$ a linear map

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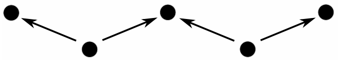

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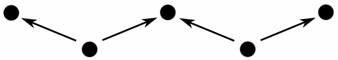

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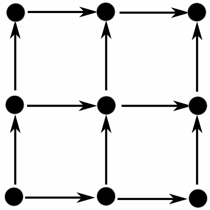

2-parameter

- a vertex $\leftarrow$ a vector space
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- Every closed subdiagram commutes.

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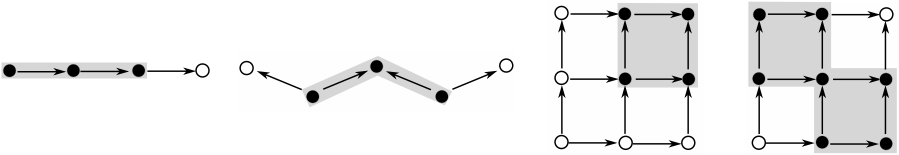
Examples.



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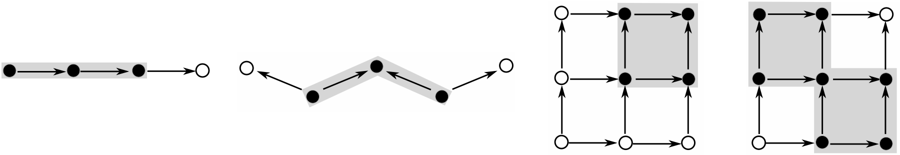

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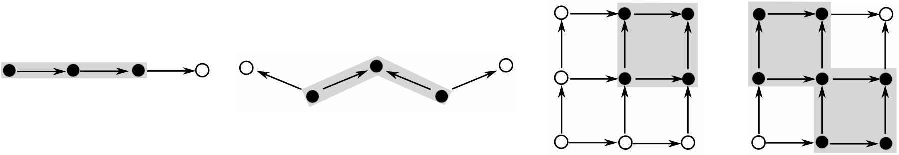

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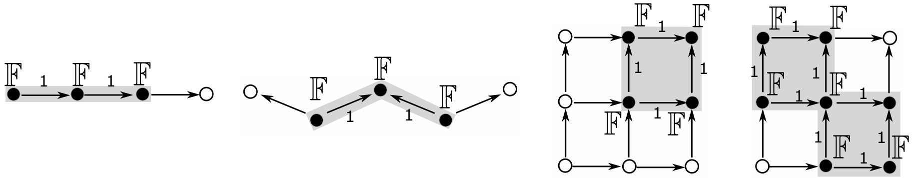

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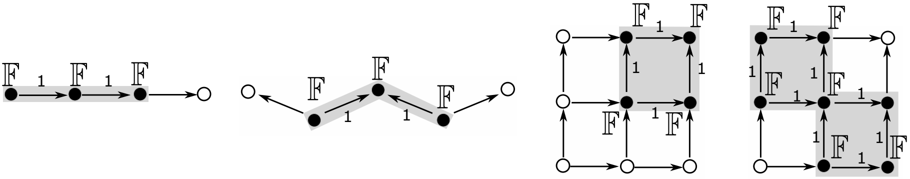

Interval modules are **indecomposable**.

Definition

If $M \cong \bigoplus_i \mathbb{I}^{I_i}$, then M is called **interval decomposable**, and the multiset $\{\{I_i\}\}$ is called the **barcode** of M.

Rank over an interval $I \subset \mathbb{P}$

Given $M : \mathbb{P} \to \textbf{vec}$, and any interval $I \subset \mathbb{P}$

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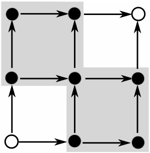

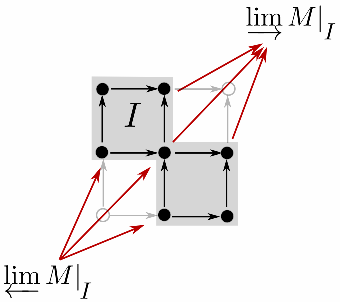

Int($\mathbb{P}$): the set of all intervals of $\mathbb{P}$.

(**Int**-)Generalized rank invariant of $M : \mathbb{P} \to \textbf{vec}$ [K,Mémoli], [Asashiba et al.]

$$\begin{aligned}\textbf{rk}_M : \textbf{Int}(\mathbb{P}) &\to \mathbb{Z}_{\geq 0}\\ I &\mapsto \textbf{rk}_M(I)\end{aligned}$$

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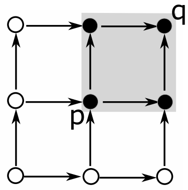

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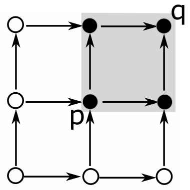

2. $\textbf{rk}_M$ is non-increasing, i.e.

$$I \subset J \;\Rightarrow\; \textbf{rk}_M(I) \geq \textbf{rk}_M(J).$$

Suppose that M is interval decomposable.

Proposition

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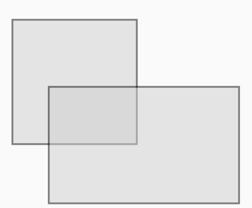
$M \cong$ 

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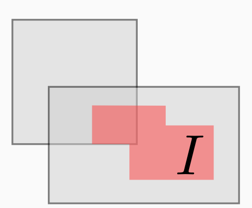
$M \cong$ 

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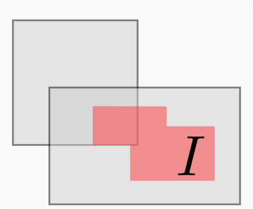
$M \cong$ 

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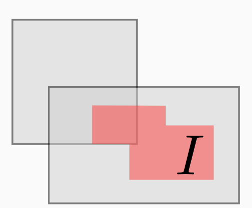
$M \cong$ 

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Via the inclusion-exclusion principle (or Möbius inversion), the multiplicity of each interval in $\mathbf{barc}(M)$ can be computed. (i.e. $\mathbf{rk}_M$ is a complete inv. of int. decomp. modules.)

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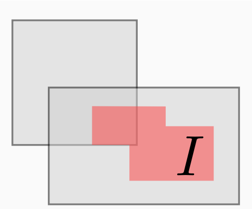
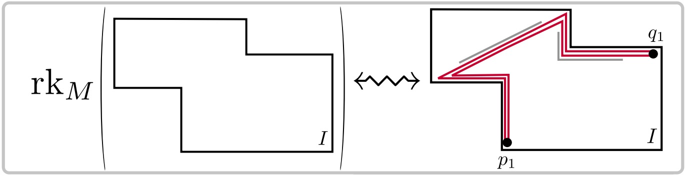
$M \cong$ 

$\mathbf{rk}_M(I) = 1.$

Via the inclusion-exclusion principle (or Möbius inversion), the multiplicity of each interval in $\mathbf{barc}(M)$ can be computed. (i.e. $\mathbf{rk}_M$ is a complete inv. of int. decomp. modules.)

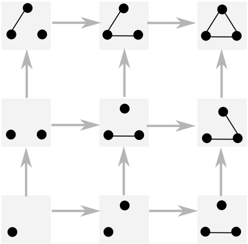
The Möbius inversion of $\mathbf{rk}_M$ yields:

1. Generalized persistence diagram of M. [K,Mémoli], [Patel]
2. Interval approximation of M. [Asashiba et al.]
3. Rank decomposition of M. [Botnan, Oppermann, Oudot]

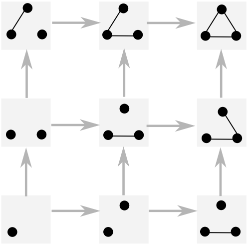


Applications

Input. A simplicial filtration $\mathcal{F}$ over a finite 2d-grid.

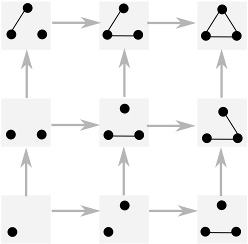


Input. A simplicial filtration $\mathcal{F}$ over a finite 2d-grid.



$M := \mathrm{H}(\mathcal{F})$

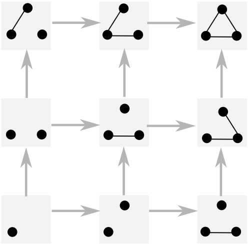
Input. A simplicial filtration $\mathcal{F}$ over a finite 2d-grid.



$M := \mathrm{H}(\mathcal{F})$

Algorithm 1. Given that M is interval decomposable, compute the barcode of M.

Input. A simplicial filtration $\mathcal{F}$ over a finite 2d-grid.



$M := \mathrm{H}(\mathcal{F})$

Algorithm 1. Given that M is interval decomposable, compute the barcode of M.

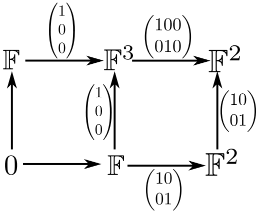
Algorithm 2. Check whether M is interval decomposable or not.

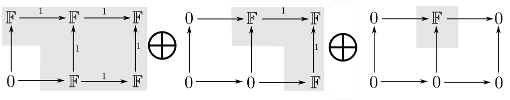
Algorithm 1. Compute the the barcode of M.

$$
M \;=\; \begin{array}{ccccc}
\mathbb{F} & \xrightarrow{\begin{pmatrix}1\\0\\0\end{pmatrix}} & \mathbb{F}^3 & \xrightarrow{\begin{pmatrix}100\\010\end{pmatrix}} & \mathbb{F}^2 \\
\uparrow & & \uparrow{\scriptstyle\begin{pmatrix}1\\0\\0\end{pmatrix}} & & \uparrow{\scriptstyle\begin{pmatrix}10\\01\end{pmatrix}} \\
0 & \longrightarrow & \mathbb{F} & \xrightarrow[\begin{pmatrix}10\\01\end{pmatrix}]{} & \mathbb{F}^2
\end{array}
$$

Algorithm 1. Compute the the barcode of M.

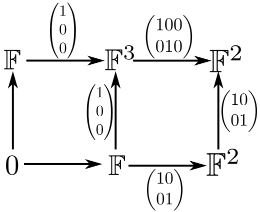
$M =$

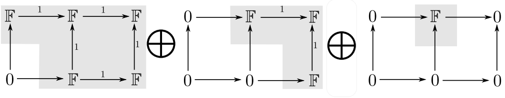




Algorithm 1. Compute the the barcode of M.

$M =$

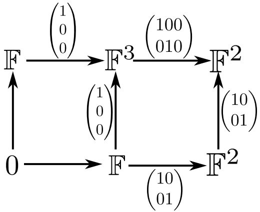


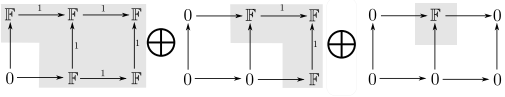


$\overrightarrow{\dim}(M) =$

1	3	2
	1	2

Algorithm 1. Compute the the barcode of M.

$M =$ 



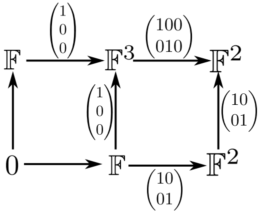
$\overrightarrow{\dim}(M) =$

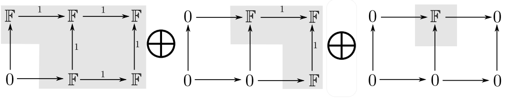
1	3	2
	1	2

Step 1. (Extension process until $\mathbf{rk}_M = 0$)

Algorithm 1. Compute the the barcode of M.

$M =$

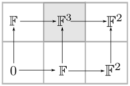




$\overrightarrow{\dim}(M) =$

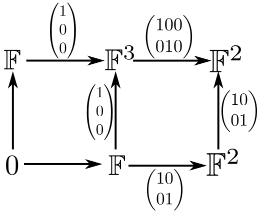
1	3	2
	1	2

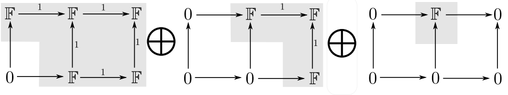
Step 1. (Extension process until $\mathbf{rk}_M = 0$)



$\mathbf{rk}_M(I) = 3$

Algorithm 1. Compute the the barcode of M.

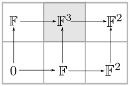
$M =$ 



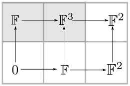
$\overrightarrow{\dim}(M) =$

1	3	2
	1	2

Step 1. (Extension process until $\mathbf{rk}_M = 0$)

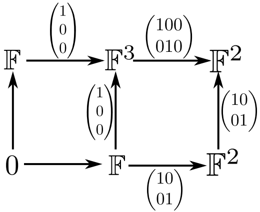


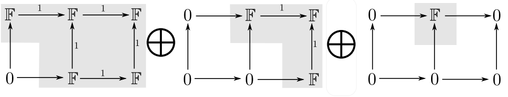
$\mathbf{rk}_M(I) = 3$



$\mathbf{rk}_M(I) = 1$

Algorithm 1. Compute the the barcode of M.

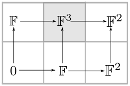
$M =$ 



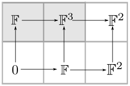
$\overrightarrow{\dim}(M) =$

1	3	2
	1	2

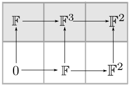
Step 1. (Extension process until $\mathbf{rk}_M = 0$)



$\mathbf{rk}_M(I) = 3$

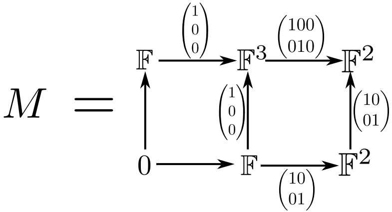


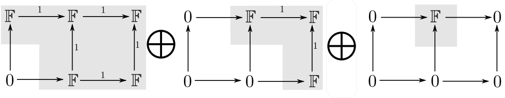
$\mathbf{rk}_M(I) = 1$



$\mathbf{rk}_M(I) = 1$

Algorithm 1. Compute the the barcode of M.

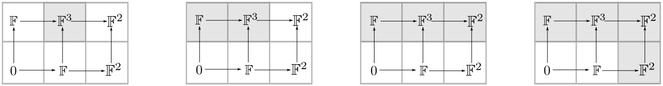




$\overrightarrow{\dim}(M) =$

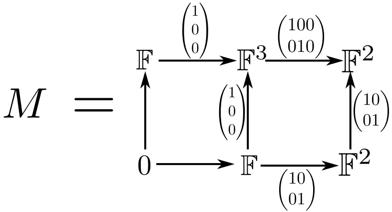
1	3	2
	1	2

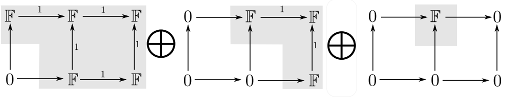
Step 1. (Extension process until $\mathbf{rk}_M = 0$)



$\mathbf{rk}_M(I) = 3$ $\quad$ $\mathbf{rk}_M(I) = 1$ $\quad$ $\mathbf{rk}_M(I) = 1$ $\quad$ $\mathbf{rk}_M(I) = 1$

Algorithm 1. Compute the the barcode of M.

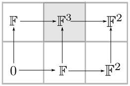




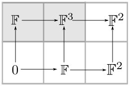
$\overrightarrow{\dim}(M) =$

1	3	2
	1	2

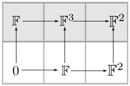
Step 1. (Extension process until $\mathbf{rk}_M = 0$)



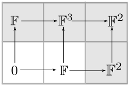
$\mathbf{rk}_M(I) = 3$



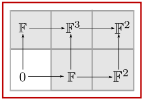
$\mathbf{rk}_M(I) = 1$



$\mathbf{rk}_M(I) = 1$

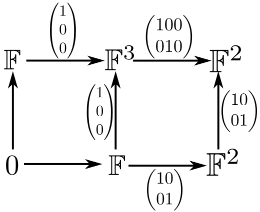


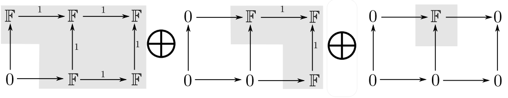
$\mathbf{rk}_M(I) = 1$



$\mathbf{rk}_M(I) = 1$

Algorithm 1. Compute the the barcode of M.

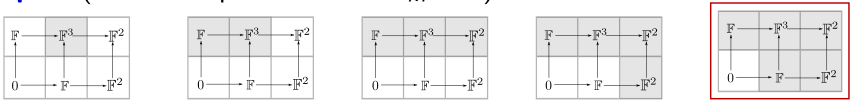
$M =$ 



$\overrightarrow{\dim}(M) =$

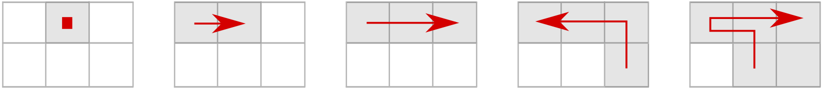
1	3	2
	1	2

Step 1. (Extension process until $\mathbf{rk}_M = 0$)



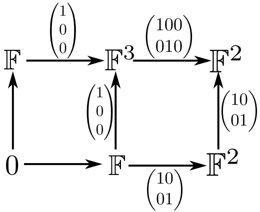
$\mathbf{rk}_M(I) = 3$ $\mathbf{rk}_M(I) = 1$ $\mathbf{rk}_M(I) = 1$ $\mathbf{rk}_M(I) = 1$ $\mathbf{rk}_M(I) = 1$

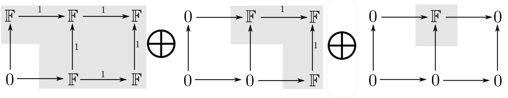
Remark. We will be using a zigzag persistence algorithm in this process.



[Milosavljevic, Morozov, Skraba 2011], [Dey, Tou 2022]

Algorithm 1. Compute the the barcode of M.

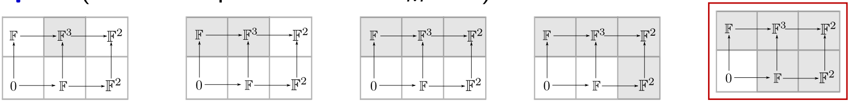
$M =$ 



$\overrightarrow{\dim}(M) =$

1	3	2
	1	2

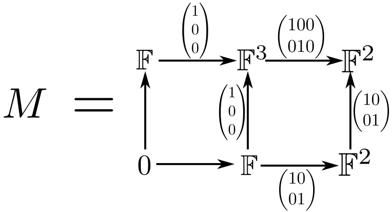
Step 1. (Extension process until $\mathbf{rk}_M = 0$)

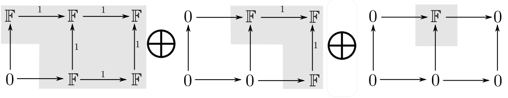


$\mathbf{rk}_M(I) = 1$

Step 2. (Subtraction)

Algorithm 1. Compute the the barcode of M.

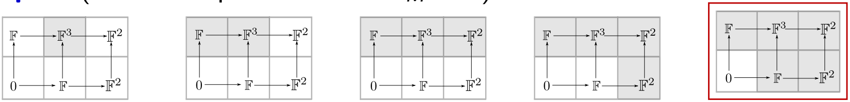




$\overrightarrow{\dim}(M) =$

1	3	2
	1	2

Step 1. (Extension process until $\mathbf{rk}_M = 0$)



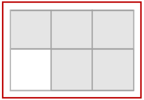
$\mathbf{rk}_M(I) = 1$

Step 2. (Subtraction)

$\overrightarrow{\dim}(M) =$

1	3	2
	1	2

$-$ $1\bullet$

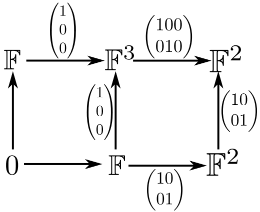


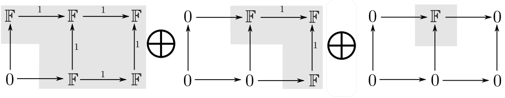
$=$

	2	1
		1

Algorithm 1. Compute the the barcode of M.

$M =$

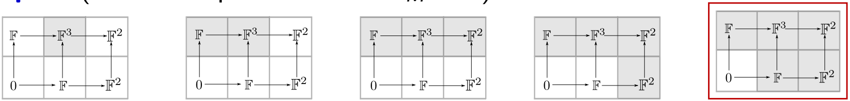




$\overrightarrow{\dim}(M) =$

1	3	2
	1	2

Step 1. (Extension process until $\mathbf{rk}_M = 0$)



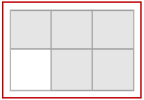
$\mathbf{rk}_M(I) = 1$

Step 2. (Subtraction)

$\overrightarrow{\dim}(M) =$

1	3	2
	1	2

$-\ 1\bullet$



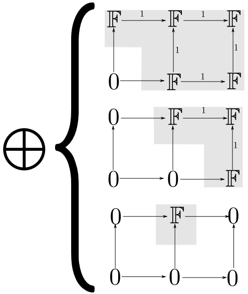
$=$

	2	1
		1

Repeat **Steps 1 and 2** until the dimension vector becomes zero.

One possible scenario:

$$M =$$



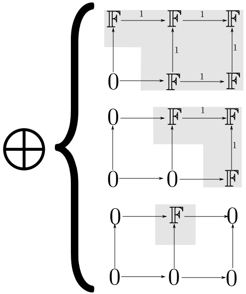
$\overrightarrow{\dim}(M)$

	2	1
		1

One possible scenario:

Step 1.

$M =$

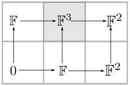


$\overrightarrow{\dim}(M)$

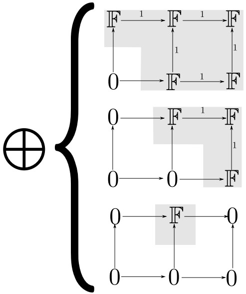
	2	1
		1

One possible scenario:

Step 1.



$$M =$$

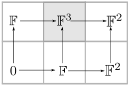


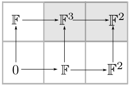
$\overrightarrow{\dim}(M)$

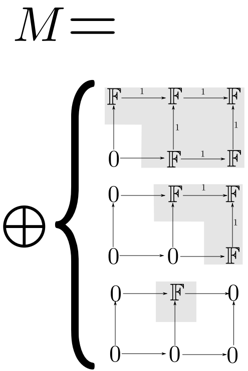
	2	1
		1

One possible scenario:

Step 1.





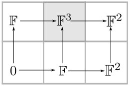


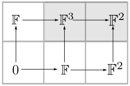
$\overrightarrow{\dim}(M)$

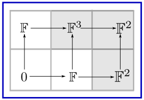
	2	1
		1

One possible scenario:

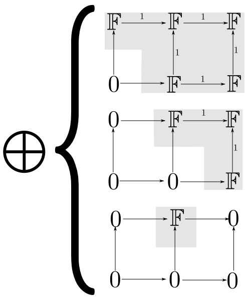
Step 1.







$M =$

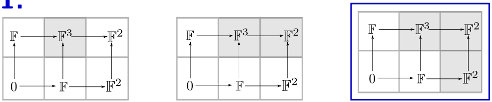


$\overrightarrow{\dim}(M)$

	2	1
		1

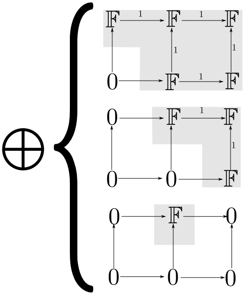
One possible scenario:

Step 1.



$\mathbf{rk}_M(I) = 2$

$M =$

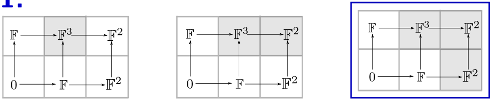


$\overrightarrow{\dim}(M)$

	2	1
		1

One possible scenario:

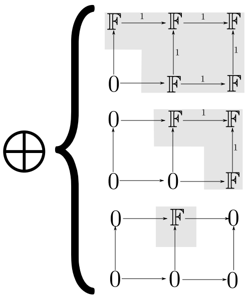
Step 1.



$\mathbf{rk}_M(I) = 2$

Step 2.

$M =$



$\overrightarrow{\dim}(M)$

	2	1
		1

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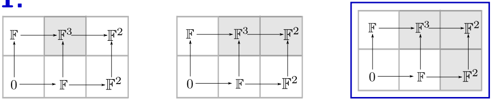
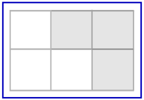


$\mathbf{rk}_M(I) = 2$

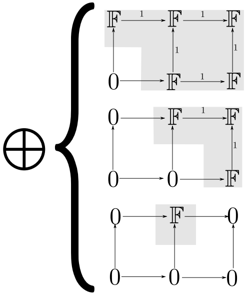
Step 2.

	2	1
		1

— 2•



$M = \oplus$



$\overrightarrow{\dim}(M)$

	2	1
		1

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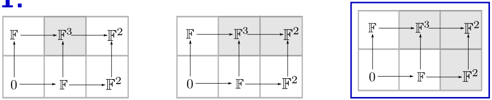

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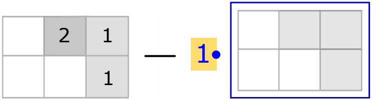

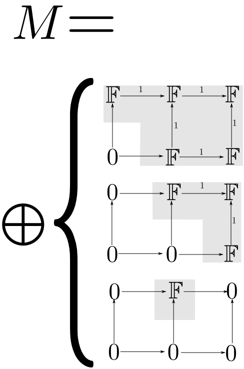

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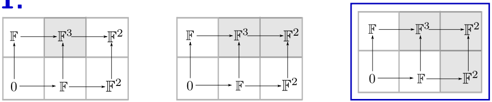

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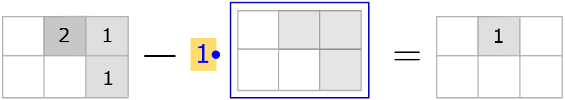

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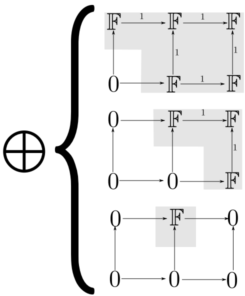

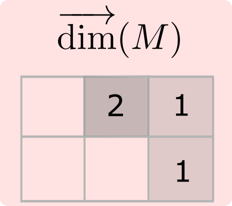

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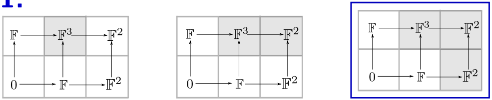

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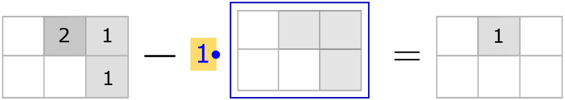

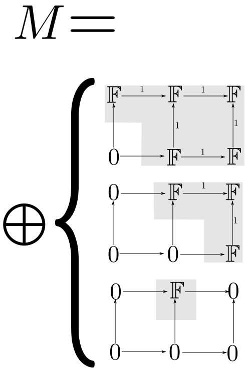

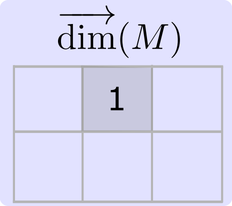

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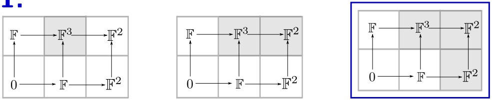

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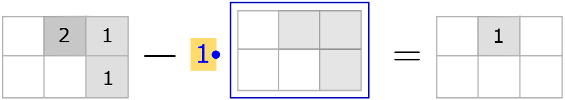

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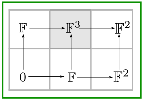

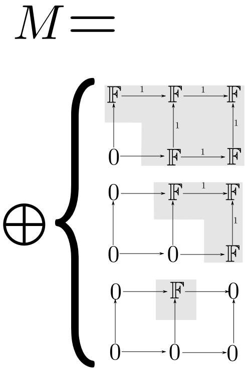

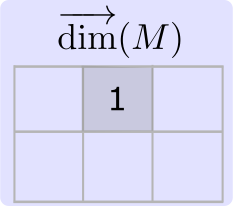

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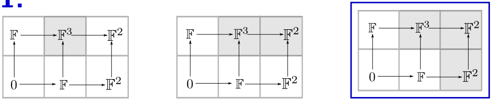

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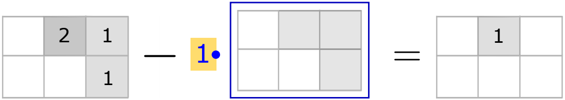

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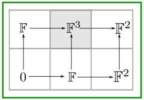
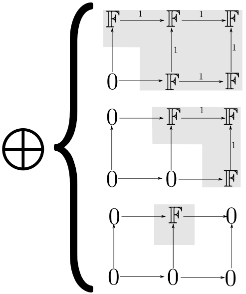


$\mathbf{rk}_M(I) = 3$

Step 2. $\overrightarrow{\dim}(M) - 1 \cdot \overrightarrow{\dim}(\mathbb{I}^I) = 0.$

$M =$



$\overrightarrow{\dim}(M)$

	1	

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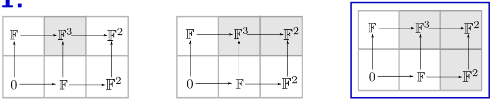
Step 1.



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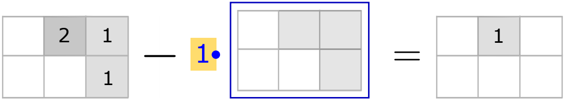

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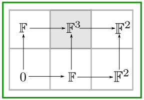

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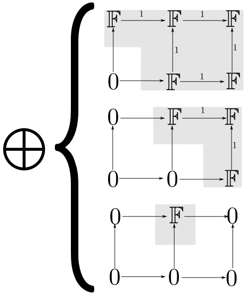

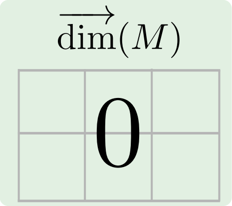

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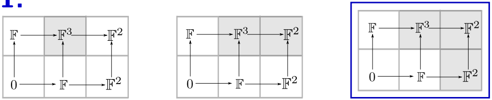

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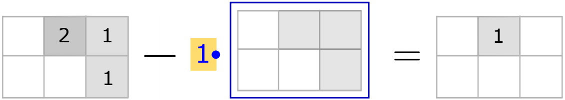

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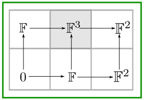

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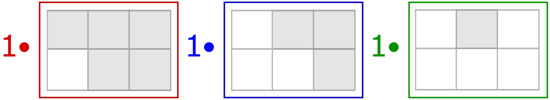
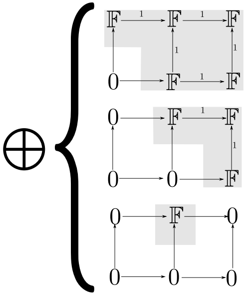
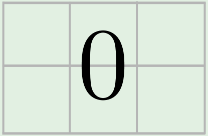
Output:



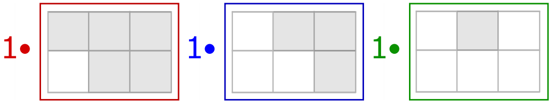
$M =$



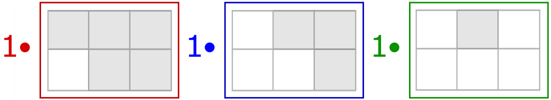
$\overrightarrow{\dim}(M)$



Output:



Output:



> **Theorem 2 (Dey, K, Mémoli)**
>
> If M is interval decomposable, then the output is $\mathbf{barc}(M)$.

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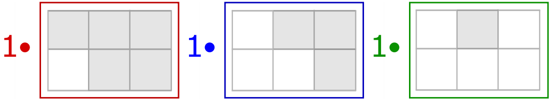


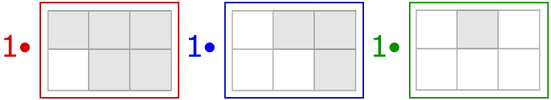
Theorem 2 (Dey, K, Mémoli)

If M is interval decomposable, then the output is $\mathbf{barc}(M)$.

Corollary

If M does not admit the output as its barcode, then M is not interval decomposable.

Output:



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Algorithm 2. Check whether M is interval decomposable.

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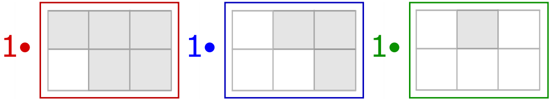

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Check if each interval I in the output is a summand of M with correct multiplicity, using **[Asashiba et al. 2021]**-algorithm.

Computational complexity

Input: 2-parameter simplicial filtration $\mathcal{F}$.

Let $t := \max\{\#\text{ of points in the indexing poset}, \#\text{ of simplices}\}$.

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- **[Asashiba et al. 2021]** requires time exponential in t, since there are exponentially many intervals.

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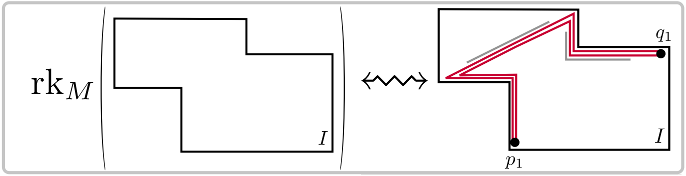
- **[Asashiba et al. 2021]** requires time exponential in t, since there are exponentially many intervals.
- **[Dey, Xin 2022]** requires only $O(t^{2\omega+1})$ time. However, $\mathrm{H}(\mathcal{F})$ must be such that no two generators or relations have the same grade.

Future work



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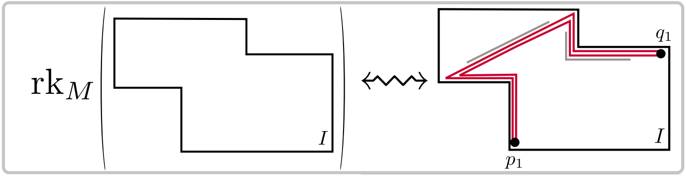

- Beyond 2-parameter persistence. (Ongoing work).

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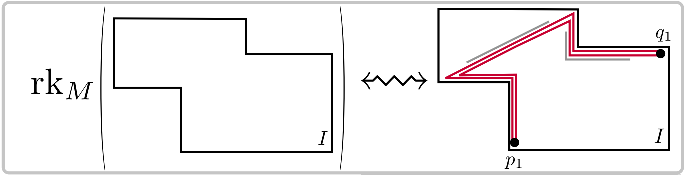

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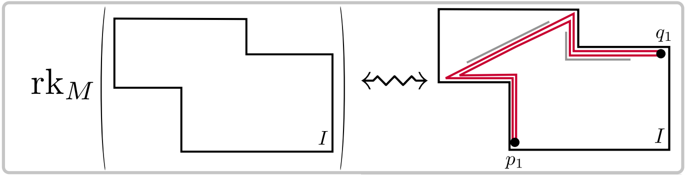

- Beyond 2-parameter persistence. (Ongoing work).
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- Improvement of the interval decomposability algorithm.

Thank you for paying attention.

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Given that $\mathrm{H}(\mathcal{F})$ is interval decomposable, the barcode can be computed in $O(t^{\omega+2})$ time.

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Comparison

$O(t^{2\omega+1})$ using **[Dey, Xin 2022]**-algorithm.

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Comparison

$O(t^{2\omega+1})$ using **[Dey, Xin 2022]**-algorithm.

(Note: **[Dey, Xin 2022]**-algorithm is more general).

Proposition 2

The interval decomposability of $\mathrm{H}(\mathcal{F})$ can be determined in $O(t^{3\omega+2})$.

1. Run **Algorithm 1**: $O(t^{\omega+2})$.

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1. Run **Algorithm 1**: $O(t^{\omega+2})$.
2. The output contains $O(t^2)$ intervals.
3. For each interval I in the barcode candidate, check whether

$$\underbrace{\mathbb{I}^I \oplus \mathbb{I}^I \oplus \mathbb{I}^I \oplus \cdots \oplus \mathbb{I}^I}_{\mu_I}$$

is the summand of M using results by [Asashiba, Buchet, Escholar, Nakashima, Yoshiwaki 2021]. It takes $O(t^{3\omega})$ in time.

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is the summand of M using results by [Asashiba, Buchet, Escholar, Nakashima, Yoshiwaki 2021]. It takes $O(t^{3\omega})$ in time.

In sum, ① + (② × ③) $= O(t^{3\omega+2})$.

Proposition 2

The interval decomposability of $\mathrm{H}(\mathcal{F})$ can be determined in $O(t^{3\omega+2})$.

1. Run **Algorithm 1**: $O(t^{\omega+2})$.
2. The output contains $O(t^2)$ intervals.
3. For each interval I in the barcode candidate, check whether

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- **[Asashiba et al. 2021]** requires time exponential in t, since there are exponentially many intervals.

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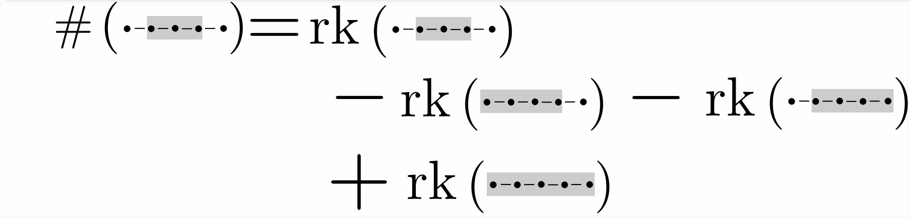
Comparison

- **[Asashiba et al. 2021]** requires time exponential in t, since there are exponentially many intervals.
- **[Dey, Xin 2019]** requires only $O(t^{2\omega+1})$ time. However, $\mathrm{H}(\mathcal{K})$ must be such that no two generators or relations have the same grade.

How to compute the barcode?


#
=
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− rk
+ rk

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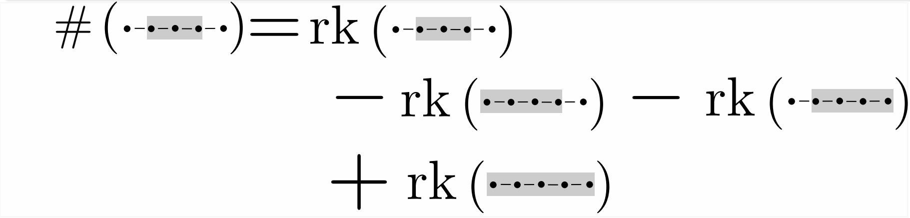
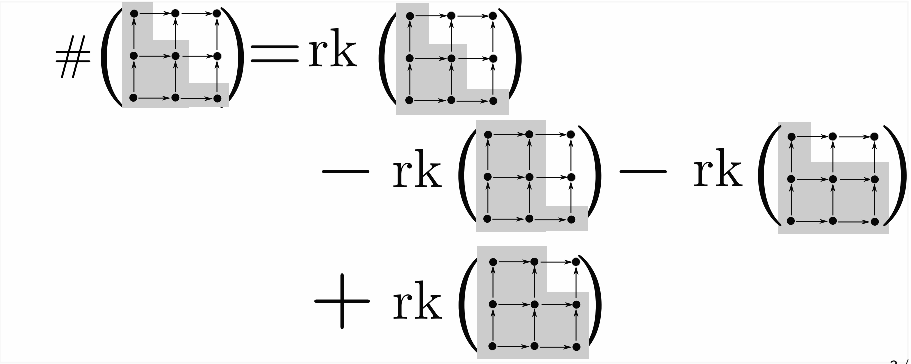
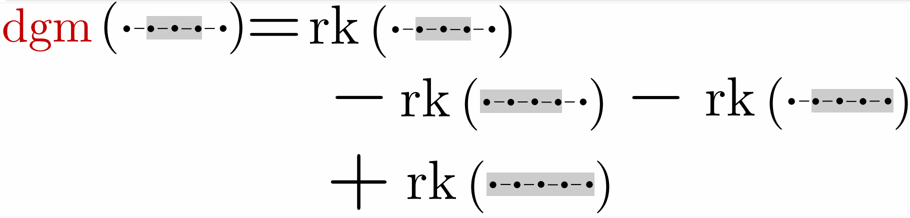
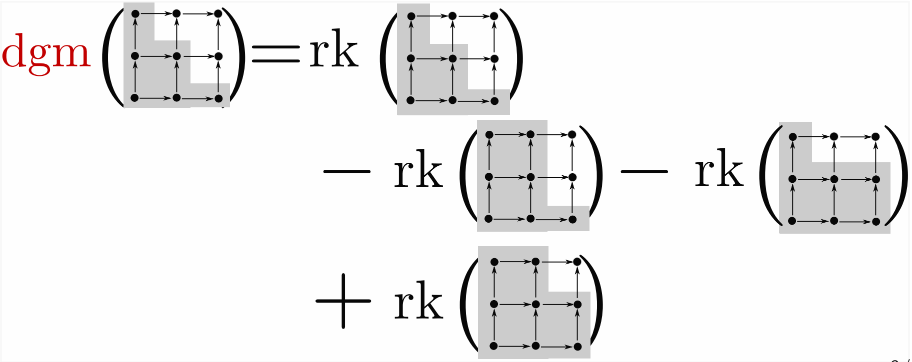


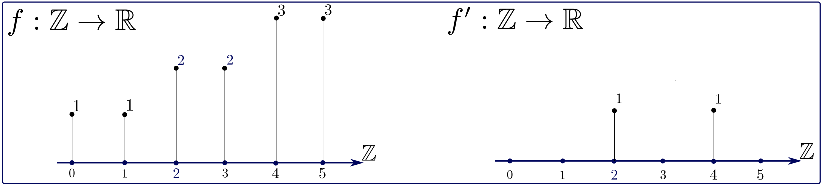


How to define the generalized persistence diagram?

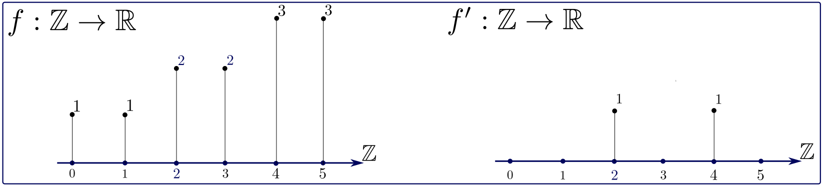




Möbius Inversion: discrete derivative



Möbius Inversion: discrete derivative



Poset	A given function	Möbius inversion
$(\mathbb{Z}, \leq)$	$f : \mathbb{Z} \to \mathbb{R}$	$f'(n) = f(n) - f(n-1)$

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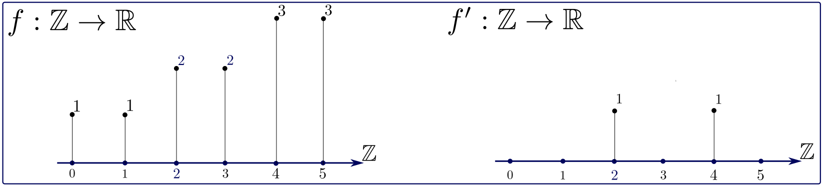

Poset	A given function	Möbius inversion
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$(\mathbf{Int}(\mathbb{P}), \supseteq)$	$\mathbf{rk}_M : \mathbf{Int}(\mathbb{P}) \to \mathbb{Z}$	$\mathbf{dgm}_M : \mathbf{Int}(\mathbb{P}) \to \mathbb{Z}$