Extracting Persistent Clusters in Dynamic Data via Möbius Inversion.

https://arxiv.org/abs/1712.04064 (updated in Feb, 2022)

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Motivation

Topological study of a collection of moving entities (e.g. collective behaviors)
Identify groups or describe the evolution of groups in time-varying metric data

Wiratma, van Kreveld, 2019

Encyclop dia Britannica

Related work (incomplete).

Buchin, Buchin, van Kreveld, Speckmann, Staals (2013),
Trajectory grouping structure

Topaz, Ziegelmeier, Halverson (2015)
Topological Data Analysis of Biological Aggregation Models

Kim, Mémoli (2020)
Spatiotemporal persistent homology for dynamic metric spaces

Ciocanel, Juenemann, Dawes, McKinley (2021)
Topological data analysis approaches to uncovering the timing of ring structure onset in lamentous networks

Xian, Adams, Topaz, Ziegelmeier (2022)
Capturing dynamics of time-varying data via topology

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Sinhuber, Oullette, 2017

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Summarize time-evolving clustering information of dynamic graphs, by adapting ideas from TDA.

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Remarks.

Formigrams are *zigzag persistence*[a] of partitions.

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[b] [Kim and Mémoli 21]
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Summarize time-evolving clustering information of dynamic graphs, by adapting ideas from TDA.

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Formigrams are zigzag persistence[a] of partitions.

To define persistence clustergrams, we adapt ideas of generalized rank[b] and Möbius inversion[c].

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This pipeline is stable.

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The maximal group diagram is another "persistence diagram" $\mathcal{Z}(Z, \underline{\mathrm{Part}})$

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π_0 : Graph(X) $\to$ Subpart(X): The path-components functor.

ZZ: $0 \quad (0,1) \quad 1 \quad (1,2) \quad 2$ The zigzag poset

Def. A dynamic graph is a poset map $G_X : \mathbb{Z} \to \mathrm{Graph}(X)$.

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Proposition (K,Memoli)

“Tame” dynamic point clouds $\xrightarrow{\mathrm{VR}_1}$ Dynamic graphs.

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These motivate us to consider further summarization of formigrams.

Lattice structure of Subpart(X)

Subpart(X): The poset of all subpartitions of X.

$\bigwedge$: The greatest lower bound (The coarsest common refinement).

$\bigvee$: The least upper bound (The finest common coarsening).

Lattice structure of $\mathrm{Subpart}(X)$

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$\bigwedge$: The greatest lower bound (The coarsest common refinement).

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e.g. $X := \{x; y; z\}$.

For example, $\{x|yz\} \wedge \{xz|y\} = \{x|y|z\}$ and $\{x|yz\} \vee \{xy|z\} = \{xyz\}$.

Notation. Int (ZZ): The collection of all \intervals" ofZZ.

Notation. Int (ZZ): The collection of all \intervals" of ZZ.

Notation. $\mathrm{Int}(\mathbb{Z})$: The collection of all "intervals" of $\mathbb{Z}$.

$\langle a, b\rangle \in \mathrm{Int}(\mathbb{Z})$ will denote one of $[a, b]$, $(a, b]$, $[a, b)$, and (a, b).

Given any $\mathcal{X} : \mathbb{Z} \to \mathrm{Subpart}(X)$ and any $\langle a, b\rangle \in \mathrm{Int}(\mathbb{Z})$,

- $\bigvee_{\langle a,b\rangle} \mathcal{X}$: The coarsest common refinement over $\langle a, b\rangle$.
- $\bigwedge_{\langle a,b\rangle} \mathcal{X}$: The finest common coarsening over $\langle a, b\rangle$.

Given any $\chi : \mathbb{Z} \to \mathrm{Subpart}(X)$ and any $\langle a, b\rangle \in \mathrm{Int}(\mathbb{Z})$,

- $\bigvee_{\langle a, b\rangle} \chi$: The coarsest common refinement over $\langle a, b\rangle$.
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e.g. $X := \{x_1, x_2, x_3, x_4, x_5\}$.

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$$\bigwedge_{[1,3]} \chi = \{x_1|x_2x_3|x_4|x_5\}$$

$$\bigvee_{[1,3]} \chi = \{x_1x_2x_3|x_4x_5\}$$

We define $\mathbb{W}_{\langle a;b\rangle}^{\times}$ as the "coimage" of $V_{\langle a;b\rangle}^{\times}$ in $W_{\langle a;b\rangle}^{\times}$.

We define $\mathbb{W}_{\langle a,b\rangle}^{X}$ as the "coimage" of $V_{\langle a,b\rangle}^{X}$ in $W_{\langle a,b\rangle}^{X}$.

We define $\mathbb{W}_{\langle a,b\rangle} X$ as the "coimage" of $V_{\langle a,b\rangle} X$ in $W_{\langle a,b\rangle} X$.

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We define $\mathbb{W}_{\langle a;b\rangle} X$ as the "coimage" of $\bigvee_{\langle a;b\rangle} X$ in $W_{\langle a;b\rangle} X$.

"coimage" in a category, different from Subpart(X).

We define $\bigvee\!\!\bigvee_{\langle a;b\rangle} X$ as the "coimage" of $\bigwedge_{\langle a;b\rangle} X$ in $\bigvee_{\langle a;b\rangle} X$.

"coimage" in a category, different from Subpart(X).

Remark. If $\bigwedge_{\langle a;b\rangle} X$ and $\bigvee_{\langle a;b\rangle} X$ have the same underlying set, then $\bigvee\!\!\bigvee_{\langle a;b\rangle} X = \bigvee_{\langle a;b\rangle} X$.

We define $\widetilde{\bigvee}_{\langle a;b\rangle} X$ as the "coimage" of $\bigwedge_{\langle a;b\rangle} X$ in $\bigvee_{\langle a;b\rangle} X$.

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Remark. If $\bigwedge_{\langle a;b\rangle} X$ and $\bigvee_{\langle a;b\rangle} X$ have the same underlying set, then $\widetilde{\bigvee}_{\langle a;b\rangle} X = \bigvee_{\langle a;b\rangle} X$.

e.g.

$$\bigwedge_{[1;3]} X = \{x_1|x_2x_3|x_4|x_5\}$$

$$\overline{X}_{[1;3]} = \{x_1x_2x_3|x_4x_5\}$$

We define $\mathbb{W}_{\langle a,b\rangle} X$ as the "coimage" of $\bigvee_{\langle a,b\rangle} X$ in $\bigwedge_{\langle a,b\rangle} X$.

"coimage" in a category, different from $\mathrm{Subpart}(X)$.

Remark. If $\bigvee_{\langle a,b\rangle} X$ and $\bigwedge_{\langle a,b\rangle} X$ have the same underlying set, then $\mathbb{W}_{\langle a,b\rangle} X = \bigwedge_{\langle a,b\rangle} X$.

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$$\bigwedge_{[1,3]} X = \{x_1|x_2x_3|x_4|x_5\}$$

$$\overline{\bigwedge}_{[1,3]} X = \{x_1x_2x_3|x_4x_5\} = \hat{\overline{\bigwedge}}_{[1,3]} X$$

Definition

The rank invariant $\mathbb{W}_X$ of $_X : ZZ \to \mathrm{Subpart}(X)$ is defined as the map:

$$\langle a, b\rangle \mapsto \bigwedge \overline{}_{\langle a,b\rangle} X .$$

Definition

The rank invariant $\mathbb{W}_X$ of $_X : \mathbb{ZZ} \to \mathrm{Subpart}(X)$ is defined as the map:

$$\langle a,b \rangle \mapsto \bigwedge \overline{}_{\langle a,b \rangle} X.$$

Remark

This definition is a rendition of generalized rank invariant for zigzag persistence [K, Mémoli 2021], which is a generalization of [Patel 2018].

Definition

The rank invariant $\mathrm{rk}\,\mathcal{M}_X$ of $\mathcal{M}_X : \mathbf{ZZ} \to \mathrm{Subpart}(X)$ is defined as the map:

$$\langle a, b\rangle \mapsto \bigwedge\bigvee_{\langle a,b\rangle} \mathcal{M}_X.$$

Remark

This definition is a rendition of generalized rank invariant for zigzag persistence [K, Mémoli 2021], which is a generalization of [Patel 2018].

Definition

The persistence clustergram of $\mathcal{M}_X$ is defined as the Möbius inversion of $\mathrm{rk}\,\mathcal{M}_X$ over the poset $(\mathrm{Int}(\mathbf{ZZ}), \supseteq)$, i.e.

$$\mathrm{dgm}^{\mathcal{M}}(\mathcal{M}_X)\langle a, b\rangle := \bigwedge\bigvee_{\langle a,b\rangle} \mathcal{M}_X - \bigwedge\bigvee_{\langle a^-,b\rangle} \mathcal{M}_X - \bigwedge\bigvee_{\langle a,b^+\rangle} \mathcal{M}_X + \bigwedge\bigvee_{\langle a^-,b^+\rangle} \mathcal{M}_X.$$

Definition

The rank invariant rk_{γ_X} of $\gamma_X : \mathbf{ZZ} \to \mathrm{Subpart}(X)$ is defined as the map:

$$\langle a, b\rangle \mapsto \bigwedge \overline{\gamma_X}|_{\langle a,b\rangle}.$$

Remark

This definition is a rendition of generalized rank invariant for zigzag persistence [K, Mémoli 2021], which is a generalization of [Patel 2018].

Definition

The persistence clustergram of γ_X is defined as the Möbius inversion of rk_{γ_X} over the poset $(\mathrm{Int}(\mathbf{ZZ}), \supseteq)$, i.e.

$$\mathrm{dgm}^{\mathrm{rk}}(\gamma_X)\langle a,b\rangle := \bigwedge \overline{\gamma_X}|_{\langle a,b\rangle} - \bigwedge \overline{\gamma_X}|_{\langle a^-,b\rangle} - \bigwedge \overline{\gamma_X}|_{\langle a,b^+\rangle} + \bigwedge \overline{\gamma_X}|_{\langle a^-,b^+\rangle}.$$

$+$ and $-$: Operations in the free abelian group generated by all nonempty subsets of X.

$$\mathrm{dgm}^{\mathbb{W}}(\ell_X)\langle a;b\rangle := \mathbb{W}_{\langle a;b\rangle} X - \mathbb{W}_{\langle a^-;b\rangle} X - \mathbb{W}_{\langle a;b^+\rangle} X + \mathbb{W}_{\langle a^-;b^+\rangle} X.$$

e.g. $X := \{x_1; x_2; x_3\}$.

$$\mathrm{dgm}^{\mathbb{W}}(\ _X)\langle a;b\rangle := \mathbb{W}_{\langle a;b\rangle} X \quad \mathbb{W}_{\langle a\ ;b\rangle} X \quad \mathbb{W}_{\langle a;b^+\rangle} X + \mathbb{W}_{\langle a\ ;b^+\rangle} X\,.$$

e.g. $X := \{x_1; x_2; x_3\}$.

$$\mathrm{dgm}^{\mathbb{W}}(\phi_X)\langle a;b\rangle := \mathbb{W}_{\langle a;b\rangle}X - \mathbb{W}_{\langle a^-;b\rangle}X - \mathbb{W}_{\langle a;b^+\rangle}X + \mathbb{W}_{\langle a^-;b^+\rangle}X:$$

e.g. $X := \{x_1; x_2; x_3\}$.

$$\mathrm{dgm}^{W}(\;X)\mathrm{h}a;b\mathrm{i} := W_{\mathrm{h}a;b\mathrm{i}}\;X \quad W_{\mathrm{h}a\;\;;b\mathrm{i}}\;X \quad W_{\mathrm{h}a;b^{+}\mathrm{i}}\;X + W_{\mathrm{h}a\;\;;b^{+}\mathrm{i}}\;X:$$

e.g. $X := \mathrm{f}\,x_1;x_2;x_3\mathrm{g}.$

$$\mathrm{dgm}^{W}(\;X)(4;6) = \hat{\;}\overline{\;}_{(4;6)}\;X \quad \hat{\;}\overline{\;}_{[4;6)}\;X \quad \hat{\;}\overline{\;}_{(4;6]}\;X + \hat{\;}\overline{\;}_{[4;6]}\;X$$

$$\mathrm{dgm}^{\mathbb{W}}(\,{}_X)\langle a;b\rangle := \mathbb{W}_{\langle a;b\rangle} X \quad \mathbb{W}_{\langle a\ ;b\rangle} X \quad \mathbb{W}_{\langle a;b^+\rangle} X + \mathbb{W}_{\langle a\ ;b^+\rangle} X:$$

e.g. $X := f\,x_1; x_2; x_3 g$.

$$\mathrm{dgm}^{\mathbb{W}}(\,{}_X)(4;6) = \overset{\wedge}{\overline{}}_{(4;6)} X \quad \overset{\wedge}{\overline{}}_{[4;6)} X \quad \overset{\wedge}{\overline{}}_{(4;6]} X + \overset{\wedge}{\overline{}}_{[4;6]} X$$

$$= \quad f\,x_1 g + f\,x_2 x_3 g$$

$$\mathrm{dgm}^{\mathrm{w}}(\ \ \mathbb{X})\langle a;b\rangle := \mathbb{W}_{\langle a;b\rangle}\ \ \mathbb{X}\quad \mathbb{W}_{\langle a\ \ ;b\rangle}\ \ \mathbb{X}\quad \mathbb{W}_{\langle a;b^{+}\rangle}\ \ \mathbb{X} + \mathbb{W}_{\langle a\ \ ;b^{+}\rangle}\ \ \mathbb{X}:$$

e.g. $\mathbb{X} := \{x_1; x_2; x_3\}$.

$$\mathrm{dgm}^{\mathrm{w}}(\ \ \mathbb{X})(4;6) = \hat{\overline{\ }}_{(4;6)}\ \ \mathbb{X}\quad \hat{\overline{\ }}_{[4;6)}\ \ \mathbb{X}\quad \hat{\overline{\ }}_{(4;6]}\ \ \mathbb{X} + \hat{\overline{\ }}_{[4;6]}\ \ \mathbb{X}$$

$$= \{x_1\} + \{x_2x_3\}\quad \{\quad x_1x_2x_3\}$$

$$\mathrm{dgm}^{w}(\ _{X})\langle a;b\rangle := \bigvee_{\langle a;b\rangle} X \quad \bigvee_{\langle a\ ;b\rangle} X \quad \bigvee_{\langle a;b^{+}\rangle} X + \bigvee_{\langle a\ ;b^{+}\rangle} X.$$

e.g. $X := \{x_1; x_2; x_3\}$.

$$\mathrm{dgm}^{w}(\ _{X})(4;6) = \wedge\!-_{(4;6)} X \quad \wedge\!-_{[4;6)} X \quad \wedge\!-_{(4;6]} X + \wedge\!-_{[4;6]} X$$

$$= \{x_1\} + \{x_2x_3\} \quad \{x_1x_2x_3\} \quad \{x_1x_2x_3\}$$

$$\mathrm{dgm}^{\mathbb{W}}(\ell_X)\langle a;b\rangle := \mathbb{W}_{\langle a;b\rangle}X - \mathbb{W}_{\langle a^-;b\rangle}X - \mathbb{W}_{\langle a;b^+\rangle}X + \mathbb{W}_{\langle a^-;b^+\rangle}X.$$

e.g. $X := \{x_1; x_2; x_3\}$.

$$\mathrm{dgm}^{\mathbb{W}}(\ell_X)(4;6) = \overline{\bigwedge}_{(4;6)}X - \overline{\bigwedge}_{[4;6)}X - \overline{\bigwedge}_{(4;6]}X + \overline{\bigwedge}_{[4;6]}X$$

$$= \{x_1\} + \{x_2x_3\} - \{x_1x_2x_3\} - \{x_1x_2x_3\} + \{x_1x_2x_3\}$$

$$\mathrm{dgm}^{w}(\mathcal{X})\langle a;b\rangle := \mathbb{W}_{\langle a;b\rangle}\mathcal{X} \;\; \mathbb{W}_{\langle a\;;b\rangle}\mathcal{X} \;\; \mathbb{W}_{\langle a;b^{+}\rangle}\mathcal{X} + \mathbb{W}_{\langle a\;;b^{+}\rangle}\mathcal{X}:$$

e.g. $\mathcal{X} := \{x_1; x_2; x_3\}$.

$$\mathrm{dgm}^{w}(\mathcal{X})(4;6) = \bigwedge\overline{}_{(4;6)}\mathcal{X} \;\; \bigwedge\overline{}_{[4;6)}\mathcal{X} \;\; \bigwedge\overline{}_{(4;6]}\mathcal{X} + \bigwedge\overline{}_{[4;6]}\mathcal{X}$$

$$= \{x_1\} + \{x_2x_3\} \;\; \{\;\; x_1x_2x_3\} \;\; \mathcal{X}\{x_1\mathcal{X}x_2\mathcal{X}x_3\} + \mathcal{X}\{x_1\mathcal{X}x_2\mathcal{X}x_3\}$$

$$= \{x_1\} + \{x_2x_3\} \;\; \{\;\; x_1x_2x_3\} \;\; (= \{x_1|x_2x_3\} \;\; \{\;\; x_1x_2x_3\}):$$

$$\mathrm{dgm}^{w}(\;_X)\langle a;b\rangle := W_{\langle a;b\rangle}\; X \quad W_{\langle a\;;b\rangle}\; X \quad W_{\langle a;b^+\rangle}\; X + W_{\langle a\;;b^+\rangle}\; X.$$

e.g. $X := \{x_1; x_2; x_3\}$.

$$\mathrm{dgm}^{w}(\;_X)(4;6) = \hat{}\!-_{(4;6)}\; X \quad \hat{}\!-_{[4;6)}\; X \quad \hat{}\!-_{(4;6]}\; X + \hat{}\!-_{[4;6]}\; X$$

$$= \{x_1\} + \{x_2x_3\} \quad \{\; x_1x_2x_3\} \; X\{x_1x_2x_3\} + X\{x_1x_2x_3\}$$

$$= \{x_1\} + \{x_2x_3\} \quad \{\; x_1x_2x_3\} \; (= \{x_1|x_2x_3\} \quad \{\; x_1x_2x_3\}).$$

$$\mathrm{dgm}^{W}(X)\langle a; b\rangle := W_{\langle a;b\rangle} X - W_{\langle a^-;b\rangle} X - W_{\langle a;b^+\rangle} X + W_{\langle a^-;b^+\rangle} X.$$

e.g. $X := \{x_1; x_2; x_3\}$.

Other than $[0; 2)$, $(4; 6)$ and $[0; 6]$, $\mathrm{dgm}^{W}(X)$ vanishes.

$$\begin{aligned}
\mathrm{dgm}^{W}(X)(4;6) &= \overline{\wedge}_{(4;6)} X - \overline{\wedge}_{[4;6)} X - \overline{\wedge}_{(4;6]} X + \overline{\wedge}_{[4;6]} X \\
&= \{x_1\} + \{x_2x_3\} - \{x_1x_2x_3\} - \{x_1x_2x_3\} + \{x_1x_2x_3\} \\
&= \{x_1\} + \{x_2x_3\} - \{x_1x_2x_3\} \quad (= \{x_1 | x_2x_3\} - \{x_1x_2x_3\}).
\end{aligned}$$

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$$ {}_X(t) = \sum_{I \ni t} \mathrm{dgm}^{w}({}_X)(I) \quad \forall t \in \mathbb{R}. $$

Remark. $_X$ can be recovered from $\mathrm{dgm}(_X)$.

$$_X(t) = \sum_{I \ni t} \mathrm{dgm}(_X)(I) \quad \forall t \in \mathbb{R}:$$

e.g.

$$_X(2) = \{x_1 | x_2 | x_3\}$$

Remark. $_X$ can be recovered from $\mathrm{dgm}^{w}(\ _X)$.

$$_X(t) = \sum_{I \ni t} \mathrm{dgm}^{w}(\ _X)(I) \quad \forall t \in \mathbb{R}.$$

e.g.

$$\begin{aligned} _X(2) &= \{x_1 | x_2 | x_3\} \\ &= \mathrm{dgm}^{w}(\ _X)[1,3) \\ &\quad + \mathrm{dgm}^{w}(\ _X)[1,5) \\ &\quad + \mathrm{dgm}^{w}(\ _X)[1,\infty) \end{aligned}$$

Remark. X can be recovered from $\mathrm{dgm}^{W}(X)$.

$$X(t) = \sum_{I \ni t} \mathrm{dgm}^{W}(X)(I) \quad \forall t \in \mathbb{R}.$$

e.g.

$$\begin{aligned} X(2) &= \{x_1 | x_2 | x_3\} \\ &= \mathrm{dgm}^{W}(X)[1;3) \\ &\quad + \mathrm{dgm}^{W}(X)[1;5) \\ &\quad + \mathrm{dgm}^{W}(X)[1;1\ ) \\ &= \{x_1 | x_2\} \quad \{ \quad x_1 x_2\} \\ &\quad + (\{x_1 x_2 | x_3\} \quad \{ \quad x_1 x_2 x_3\}) \\ &\quad + \{x_1 x_2 x_3\} \end{aligned}$$

Remark. $_X$ can be recovered from $\mathrm{dgm}^{\mathrm{w}}(_X)$.

$$_X(t) = \sum_{I \ni t} \mathrm{dgm}^{\mathrm{w}}(_X)(I) \quad \forall t \in \mathbb{R}.$$

e.g.

$$\begin{aligned} _X(2) &= \{x_1|x_2|x_3\} \\ &= \mathrm{dgm}^{\mathrm{w}}(_X)[1;3) \\ &\quad + \mathrm{dgm}^{\mathrm{w}}(_X)[1;5) \\ &\quad + \mathrm{dgm}^{\mathrm{w}}(_X)[1;1\ ) \\ &= \{x_1|x_2\} \sum \{x_1 x_2\} \\ &\quad + (\{x_1 x_2|x_3\} \quad \{x_1x_2x_3\}) \\ &\quad + \{x_1x_2x_3\} \end{aligned}$$

Remark. $_X$ can be recovered from $\mathrm{dgm}^{w}(\ _X)$.

$$_X(t) = \sum_{I \ni t} \mathrm{dgm}^{w}(\ _X)(I) \quad \forall t \in \mathbb{R}.$$

e.g.

$$
\begin{aligned}
_X(2) &= \{x_1|x_2|x_3\} \\
&= \mathrm{dgm}^{w}(\ _X)[1;3) \\
&\quad + \mathrm{dgm}^{w}(\ _X)[1;5) \\
&\quad + \mathrm{dgm}^{w}(\ _X)[1;1\ ) \\
&= \{x_1|x_2\} \textstyle\sum \{x_1 x_2\} \\
&\quad + (\textstyle\sum \{x_1 x_2|x_3\} \textstyle\sum \{x_1 x_2 x_3\}) \\
&\quad + \textstyle\sum \{x_1 x_2 x_3\}
\end{aligned}
$$

Remark. $_X$ can be recovered from $\mathrm{dgm}^{w}(_X)$.

$$ _X(t) = \sum_{I \ni t} \mathrm{dgm}^{w}(_X)(I) \quad \forall t \in \mathbb{R}: $$

e.g.

$$
\begin{aligned}
_X(2) &= \{x_1|x_2|x_3\} \\
&= \mathrm{dgm}^{w}(_X)[1;3) \\
&\quad + \mathrm{dgm}^{w}(_X)[1;5) \\
&\quad + \mathrm{dgm}^{w}(_X)[1;\infty) \\
&= \{x_1|x_2\} \oplus \{x_1 x_2\} \\
&\quad + (\{x_1 x_2|x_3\} \oplus \{x_1 x_2 x_3\}) \\
&\quad + \{x_1 x_2 x_3\}
\end{aligned}
$$

Analogous to:

$\dim M_{r_0} = \# \{ I \in \mathrm{barc}(M) : I \ni r_0 \}$

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: post-composing a functor.

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Collapse :
Subpart(X) $!$ **set**

$\#$ collapse

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Not always [Zigzag barcode = *Silhouette*].

collapse

Theorem (K, Mémoli)

Silhouettes are stable in the *erosion distance* [Patel 18].

Summary

Remarks.

Formigrams are *zigzag persistence*[a] of partitions.

To define **persistence clustergrams**, we adapt ideas of *generalized rank*[b] and *Mobius inversion.*[c]

Formigrams $ Persistence clustergrams

This pipeline is stable.

[a] [Carlsson, de Silva 09], [Patel, Curry 20]
[b] [Kim and Mémoli 21]
[c] [Rota 64], [Patel 18]

A spin-off

Maximal groups [Buchin et al. 13] can be computed via Möbius inversion.

The *maximal group diagram* is also a complete invariant of formigrams.
The *maximal group diagram* is another "persistence diagram" (ZZ, Part)

Open questions

Classification of dynamic graphs that arise from Boids, using zigzag barcodes

In: *Topological Data Analysis, Abel Symposia*, 371-389, [K,Mémoli,Smith 20]

Devise an annotation-sensitive bottleneck distance.

Thank you for paying attention!

Extracting Persistent Clusters in Dynamic Data via Möbius inversion https://arxiv.org/abs/1712.04064

Open questions

Classification of dynamic graphs that arise from Boids, using zigzag barcodes

In: *Topological Data Analysis, Abel Symposia*, 371-389, [K,Mémoli,Smith 20]

Devise an annotation-sensitive bottleneck distance.

Find an efficient way to compute persistence clustergrams.

Thank you for paying attention!

Extracting Persistent Clusters in Dynamic Data via Möbius inversion https://arxiv.org/abs/1712.04064