Generalized Persistence Diagrams for Persistence Modules over Posets

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UFTDA

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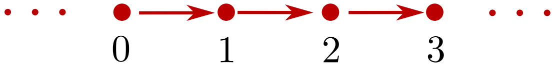

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Goal: **Use "this idea" to generalize the notion of persistence diagrams.**

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$\cdots$

Cohen−Steiner,Edelsbrunner,Harer (2007)

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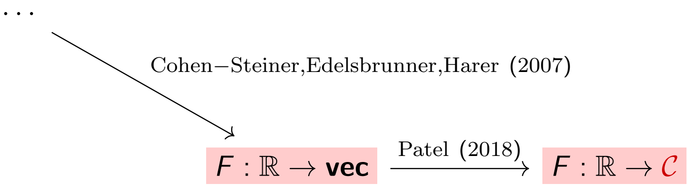

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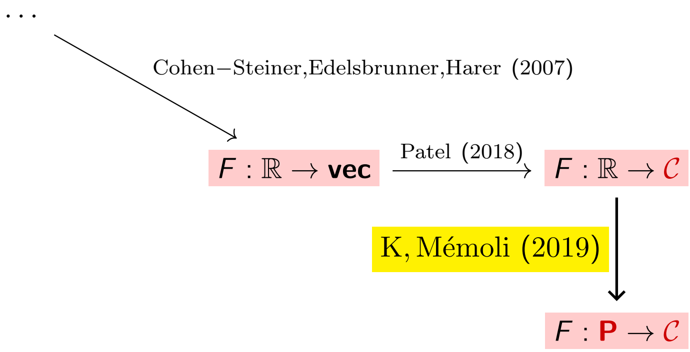

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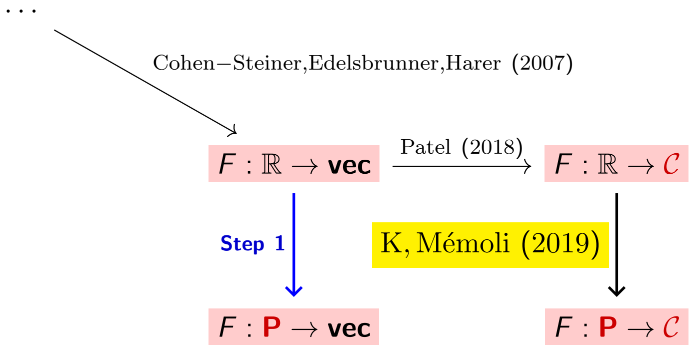

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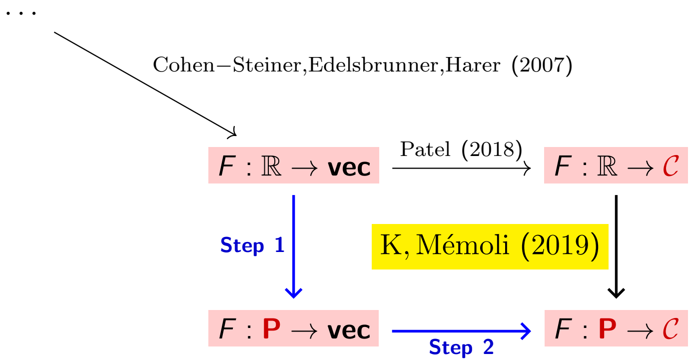

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We will define
1. Rank
2. Rank invariant
3. Persistence diagram

of $F : \mathbf{P} \to \textbf{vec}$.

Rank

Let $\mathbf{P}$ be a **connected** poset, i.e. $\forall p, q \in \mathbf{P}$ there exists a sequence $p = p_0, p_1, \ldots, p_n = q$ in $\mathbf{P}$ s.t. p_i and p_{i+1} are comparable.

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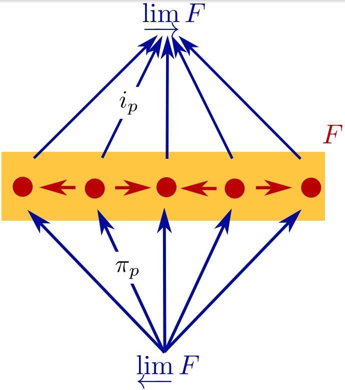
Given $F : \mathbf{P} \to \mathbf{vec}$,

$$\mathbf{rank}(F) := \mathbf{rank}\left(\phi_F : \varprojlim F \to \varinjlim F\right)$$



$\phi_F = i_p \circ \pi_p$ for any $p \in \mathbf{P}$.

Studying the Limit-to-Colimit map in TDA is **not** a new idea.

1. Work by **Patel and MacPherson** circa 2012.
2. **Chambers and Letscher** *"Persistent Homology Over Directed Acyclic Graphs" (arXived in 2014, published in 2018)*.
3. **Botnan, Oppermann and Steen**, a work in progress. See Botnan's AATRN online seminar on 04/05/2017 (available on Youtube).

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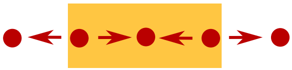

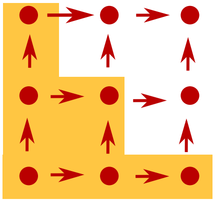

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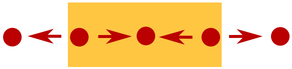

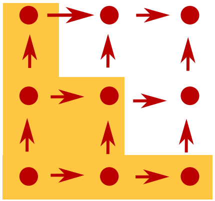

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Rank invariant of $F : \mathbf{P} \to \mathbf{vec}$

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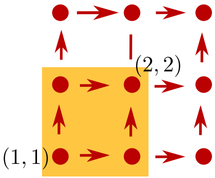

rk $F((1,1) \leq (2,2))$

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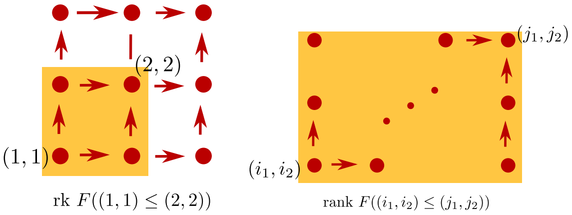

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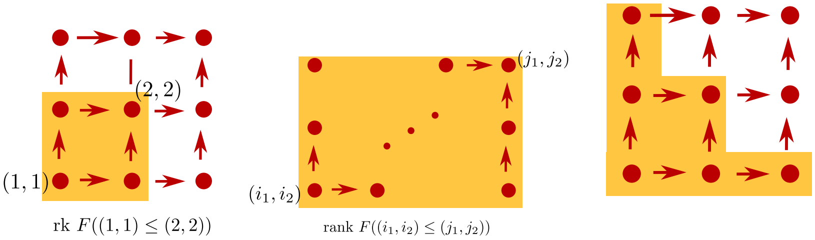

We will see that

The rank invariant is a complete invariant for **interval decomposable persistence modules.**

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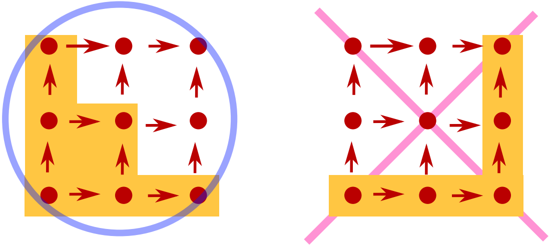

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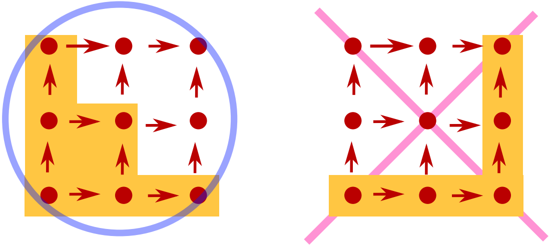

If **P** is linear (), then $\mathbf{Con}(\mathbf{P}) = \mathbf{Int}(\mathbf{P})$.

Let $K \in \mathbf{Int}(\mathbf{P})$. The interval module $I^K : \mathbf{P} \to \mathbf{vec}$ is defined as

$$I^K(p) = \begin{cases} k, & p \in K \\ 0, & \text{otherwise,} \end{cases} \qquad I^K(p \leq q) = \begin{cases} id_k, & p, q \in K \\ 0, & \text{otherwise.} \end{cases}$$

Definition

$F : \mathbf{P} \to \mathbf{vec}$ is called **interval decomposable** if there exists a multiset $\mathbf{barc}(F)$ of intervals of $\mathbf{P}$ such that

$$F \cong \bigoplus_{K \in \mathbf{barc}(F)} I^K.$$

Proposition 1 (K,Mémoli 19)

Assume that F is interval decomposable. For $K \in \mathbf{Con}(\mathbf{P})$,

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Using this proposition, we will compute $\mathbf{barc}(F)$ from $\mathbf{rk}_F$.

Let $K \in \mathbf{Con}(\mathbf{P})$.

Definition (Neighborhood of K)

$$\mathbf{nbd}(K) := \{p \in \mathbf{P} \setminus K : \exists q \in K,\ p \text{ and } q \text{ are adjacent in } \overline{\mathbf{Hasse}}(\mathbf{P}).\}$$

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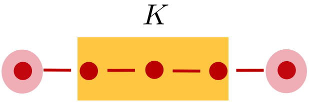
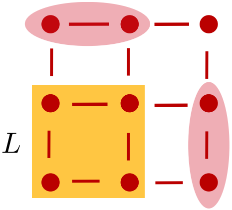


$o_K = 2$



$o_L = 4$

Definition (n-th entourage of K)

Let $K \in \mathbf{Con}(\mathbf{P})$.

$$K^n := \{L \subset \mathbf{P} : K \subset L \subset K \cup \mathbf{nbd}(K) \text{ with } |L \cap \mathbf{nbd}(K)| = n\}.$$

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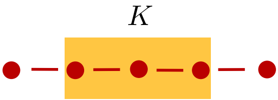

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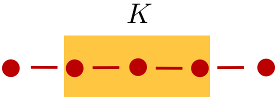

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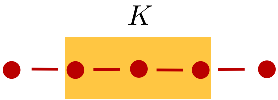

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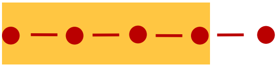

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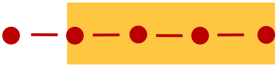

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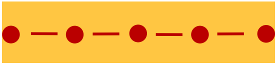

$K^n = \emptyset$ for $n > 2$.

Theorem 2 (K,Mémoli 19)

Assume that $F : \mathbf{P} \to \mathbf{vec}$ is interval decomposable. For $I \in \mathbf{Int}(\mathbf{P})$,

$$(\# \text{ of } I \text{ in } \mathbf{barc}(F)) =$$

$$\mathbf{rk}_F(I) - \left(\sum_{K \in I^1} \mathbf{rk}_F(K) - \sum_{L \in I^2} \mathbf{rk}_F(L) - \ldots + (-1)^{o_I} \sum_{M \in I^{o_I}} \mathbf{rk}_F(M) \right).$$

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Assume that $F : \mathbf{P} \to \mathbf{vec}$ is interval decomposable. For $I \in \mathbf{Int}(\mathbf{P})$,

$$(\# \text{ of } I \text{ in } \mathbf{barc}(F)) =$$

$$\mathbf{rk}_F(I) - \left(\sum_{K \in I^1} \mathbf{rk}_F(K) - \sum_{L \in I^2} \mathbf{rk}_F(L) - \ldots + (-1)^{o_I} \sum_{M \in I^{o_I}} \mathbf{rk}_F(M) \right).$$

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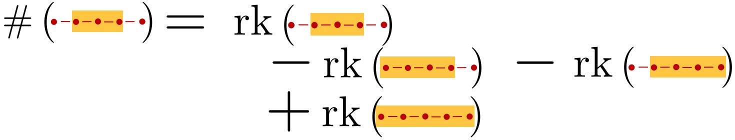
is $\# \{\!\{ J \in \mathbf{barc}(F) : I \subsetneq J \}\!\}$.

Examples



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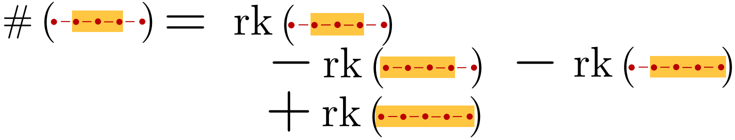

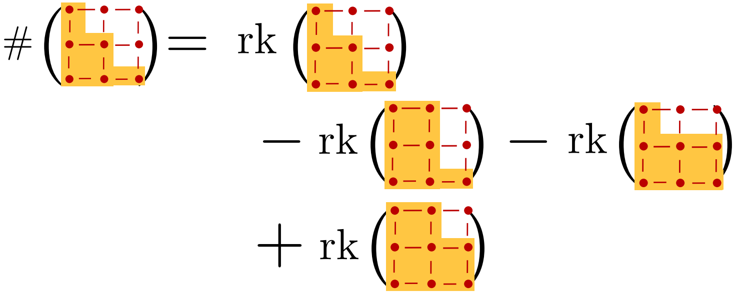

Corollary 3

The rank invariant $\mathbf{rk}_F : \mathbf{Con}(\mathbf{P}) \to \mathbb{Z}_{\geq 0}$ is a complete invariant for interval decomposable persistence modules $F : \mathbf{P} \to \mathbf{vec}$.

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Theorem 4 (K,Mémoli 19)

Let $F : \mathbf{P} \to \mathbf{vec}$. The map $\mathbf{dgm}_F : \mathbf{Con}(\mathbf{P}) \to \mathbb{Z}$ sending I to

$$\mathbf{rk}_F(I) - \sum_{K \in I^1} \mathbf{rk}_F(K) + \sum_{L \in I^2} \mathbf{rk}_F(L) + \ldots + (-1)^{o_I} \sum_{M \in K^{o_I}} \mathbf{rk}_F(M).$$

is the Möbius inversion of $\mathbf{rk}_F : \mathbf{Con}(\mathbf{P}) \to \mathbb{Z}_{\geq 0}$.

Proof idea. Compute the Möbius function of $(\mathbf{Con}(\mathbf{P}), \supset)$ – Make use of several classical results in [Rota1964].

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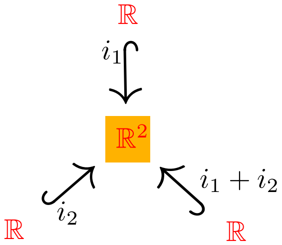

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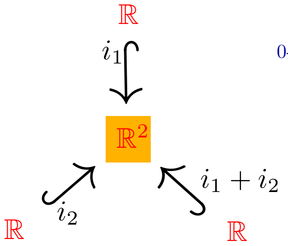

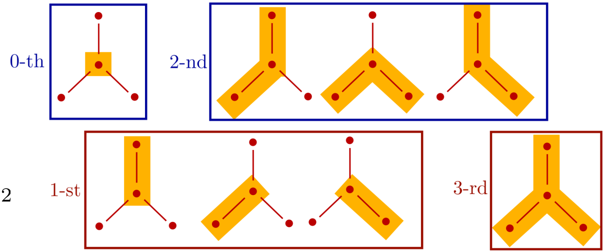

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We will adopt other ideas in [Patel18].

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Open questions and future work

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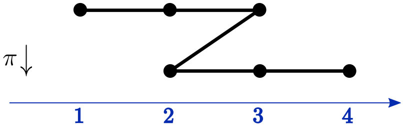
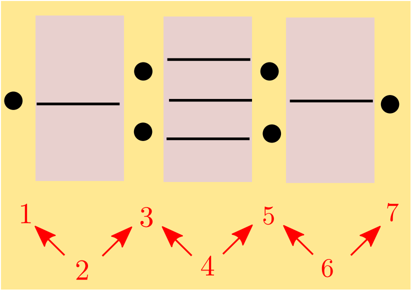
Thank you for paying attention.

https://arxiv.org/abs/1810.11517

Example.





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&= \sum_{L \in \mathbf{barc}(F)} \mathbf{rank}\left(I^L|_K\right) \\
&= \sum_{L \in \mathbf{barc}(F)} \mathbf{1}_{\{L \supset K\}}. \square
\end{aligned}$$

Möbius inversion formula

Let $\mathbf{Q}$ be a locally finite poset and let k be a field. Suppose that an element $0 \in \mathbf{Q}$ exists with the property that $0 \leq q$ for all $q \in \mathbf{Q}$. Consider a pair of functions $f, g : \mathbf{Q} \to k$ with the property that

$$g(q) = \sum_{r \leq q} f(r).$$

Then, $f(q) = \sum_{r \leq q} g(r) \cdot \mu_{\mathbf{Q}}(r, q)$ for $q \in \mathbf{Q}$.

Given $F : \mathbf{P} \to \mathbf{vec}$,

$$\mathbf{Q} = \mathbf{Con}(\mathbf{P}),\ g = \mathbf{rk}_F,\ f = \mathbf{dgm}_F.$$